Euler coordinates for infinity points not on the Euler line

Table shows X(N) infinity point on the line or the axis listed in the last column much the same as the Euler infinity point X(30) on the Euler line. The Euler coordinates for each X(N) infinity point are those for the simplest barycentrics of the triangle center X(N). Then, the triangle centers X(n) - X(m) on the line or the axis which passes through X(n) and X(m) are shown to be equivalent to X(N) infinity point on the same line or the axis, much the same as the triangle centers X(n) - X(m) on the Euler line are equivalent to the Euler infinity point X(30) (see **Tables** “Euler coordinates for X(n)-X(m) on Euler line”) .

As an example, let’s consider the triangle centers on the Brocard axis such as X(3), X(6), X(15), X(32), X(39), X(50), etc.(see **Tables** “Central line”), Then, X(n)-X(m) on the Brocard axis are shown to be equivalent to X(511) infinity point (say, “Brocard infinity point”). The Euler coordinates for X(511) infinity point are given by those for the simplest barycentrics (S_AS^2 - $(E+F)S_BS_C$ ::) of the triangle center X(511). Then, the infinity point X(511) = P_1 - P_2 is given as

$$P_1 = (S_A, S_B, S_C)S^2/\{(E+F)S^2\},$$
$$P_2 = ((E+F)S_BS_C, (E+F)S_CS_A, (E+F)S_AS_B)/\{(E+F)S^2\}.$$

The Euler coordinates for P_1 and P_2 are given by

$$x_e(P_1)=\$S_A^2\$/(E+F)=\{(E+F)^2-2S^2\}/(E+F), \quad y_e(P_1)= \$S_A^2(S_B-S_C)\$/(E+F),$$
$$x_e(P_2) = 3(E+F)F/(E+F), \quad y_e(P_2) = 0.$$

Hence, the Euler coordinates for X(511) infinity point are given by

$$x_e(511) = x_e(P_1) - x_e(P_2) = \{(E+F)(E-2F)-2S^2\}/(E+F)$$
$$y_e(511) = y_e(P_1) - y_e(P_2) = \$S_A^2(S_B-S_C)\$/(E+F).$$

as shown in the Table. Next, let’s consider the triangle center X(3) -X(6) on the Brocard axis, The triangle center X(3) -X(6) is given by

$$P_1 = X(3) = (S^2-S_BS_C)/(2S^2)$$
$$P_2 = X(6).=\{(E+F)S^2-S_AS^2\}/\{2(E+F)S^2\}$$

Then, the Euler coordinates for P_1 and P_2 are given by

$x_e(P_1) = (E-2F)\}/2,$ $\quad y_e(P_1) = 0,$
$x_e(P_2) = S^2/(E+F),$ $\quad y_e(P_2) = -\$S_A{}^2(S_B-S_C)\$/\{2(E+F)\}$

Hence, the Euler coordinates (xe(3,6), ye(3,6)) for X(3)- X(6) are given by

$x_e(3,6) = x_e(P_1) - x_e(P_2) = \{(E-2F)(E+F)-2S^2\}/\{2(E+F)$
$y_e(3,6) = y_e(P_1) - y_e(P_2) = \$S_A{}^2(S_B-S_C)\$/\{2(E+F)\}.$

Then, the triangle center X(3) - X(6) is found to be equivalent to X(511) infinity point and given by

. X(3) - X(6) = x(3, 6)X(511), x(3,6) = 1/2

To make sure the equivalence more, let’s consider the triangle center X(3)-X(15) on the Brocard axis. Then, the triangle center X(3) - X(15) is given by

$P_1 = X(3) = (S^2-S_BS_C)/(2S^2) ::$
$P_2 = X(15).=\{(E+F)S+(3)^{1/2}(S^2-S_BS_C)-S_AS\}/[2S\{(E+F)+(3)^{1/2}S\}]$

The Euler coordinates for P_1 and P_2 are given by

$x_e(P_1) = (E-2F)\}/2,$ $\quad y_e(P_1) = 0,$
$x_e(P_2) = [(3)^{1/2}(E-2F)S+2S^2]/[2S\{(E+F)+(3)^{1/2}S\}],$
$y_e(P_2) = -\$S_A{}^2(S_B-S_C)\$S/[2S\{(E+F)+(3)^{1/2}S\}]$

Hence, the Euler coordinates ($x_e(3,15)$, $y_e(3,15)$) for X(3)- X(15) are given by

$x_e(3,15) = \{(E+F)(E-2F)-2S^2\}/[\{2(E+F)+(3)^{1/2}S\}]$
$y_e(3,15) = \$S_A{}^2(S_B-S_C)\$/ [\{2(E+F)+(3)^{1/2}S\}].$

This shows that the triangle center X(3)-X(15) is equivalent to X(511) infinity point and given by

X(3) - X(15) = x(3,15)X(511), $x(3,15) = (E+F)/\{2(E+F)+(3)^{1/2}S\}$..

Similarly, the triangle centers X(n)-X(m) on the Brocard axis can be shown to be equivalent to X(511) infinity point and given by generally

$$X(n) - X(m) = x(n, m)X(511).$$

where the x(n, m) is a constant which has to be evaluated one by one as shown above.
As a conclusion, **the triangle centers X(n) - X(m) on the line or the axis are equivalent to the infinity point on the same line or the same axis**,

Finally, the line passing through X(n) = (u(n) : v(n) : w(n)) and X(m) = (u(m) : v(m) : w(m) is given by

$$[v(n)w(m) - w(n)v(m)]x+[w(n)u(m)- u(n)w(m)]y +[u(n)v(m)- v(n)u(m)]z = 0.$$

For example, the Brocard axis passing through X(3) = ($S^2 - S_BS_C : S^2 - S_CS_A : S^2 - S_AS_B$) and X(6) = ((E+F) - S_A : (E+F) - S_B : (E+F) - S_C) is given by

$$(S_A^2+ S^2)(S_B - S_C)x + (S_B^2+ S^2)(S_C - S_A)y + (S_C^2+ S^2)(S_A - S_B)z = 0.$$

Here, let's consider the line given by

$$(S_A^2+ S^2)(S_B - S_C)x+ (S_B^2+ S^2)(S_C - S_A)y+ (S_C^2+ S^2)(S_A - S_B)z = D,$$

where D is a constant. Then, the line is parallel to the Brocard axis and D is proportional to the distance from the Brocard axis. Then, the triangle centers X(n) - Xm), where X(n) and X(m) are on the line parallel to the Brocard axis, are on the Brocard axis and equivalent to X(511). This is true for lines or axes in the Table..
As a conclusion, **the triangle centers X(n) - X(m) on the line parallel to the line or the axis are equivalent to the infinity point on the line or the axis**,

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Table X(N) infinity point, Euler coordinates and Line or Axis passing through X(N) infinity point

X(N)	$x_e(n)$	$y_e(n)$	Line or Axis
X(30)	(E-8F)/3	0	Euler(L)

X(511)	$\{(E+F)(E-2F)-2S^2\}/\{(E+F)\}$	$\$S_A^2(S_B-S_C)\$/\{(E+F)\}$	Brocard(A).
X(512)	$-\$S_A^2(S_B-S_C)\$/\{(E+F)^2-2S^2\}$	$\{(E+F)(E-2F)-2S^2\}S^2$ $/\{(E+F)^2-2S^2\}$	Lemoine(A).
X(513)	$\$a(b-c)S_A\$/\$ab\$$	$\$a(b-c)S_A(S_B-S_C)\$/\$ab\$$	Anti-Orthic(A)
X(514)	$\$(b-c)S_A\$/\$a\$$	$\$(b-c)S_A(S_B-S_C)\$/\$a\$$	Gergonne(L)
X(515)	$(3F\$a\$-\$aS_A\$)/\$a\$$	$-\ \$aS_A(S_B-S_C)\$/\$a\$$	X(1)X(4)(L)
X(516)	$\{\$a\$S^2-3\$aS_BS_C\$\}$ $/\{(E+F)\$a\$-\$aS_A\$\}$	$\$a(S_B-S_C)\$S^2$ $/\{(E+F)\$a\$-\$aS_A\$\}$	Soddy(L)
X(517)	$-\{\$ab\$S^2-3\$abS_AS_B\$\}$ $/\{abc\$a\$\}$	$-\$ab(S_A-S_B)\$S^2/\{abc\$a\$\}$	OI(L)
X(518)	$\{\$a\$S^2-(E+F)\$aS_A\$\}$ $/\{(E+F)\$a\$\}$	$-\{\$S_A^2(S_B-S_C)\$\$a\$:+2(E+F)$ $\$aS_A(S_B-S_C)\$\}/\{2(E+F)\$a\$\}$	IK(L)
X(519)	$-\{(E+F)\$a\$-3\$aS_A\$\}/(3\$a\$)$	$\$aS_A(S_B-S_C)\$/\$a\$$	Nagel(L)
X(520)	$-\$S_A^2(S_B-S_C)\$/\{(E+F)F+S^2\}$	$\{(7E-2F)F-2S^2\}S^2$ $/\{(E+F)F+S^2\}$	
X(521)	$\$abS_AS_B(S_A-S_B)\$/\$abS_AS_B\$$	$[2\$abS_AS_B\$-3(E+F)F\$ab\$$ $+3\$abS_A\$F]S^2/\$abS_AS_B\$$	
X(522)	$-\$a(S_A-S_B)S_C\$/\$aS_A\$$	$-[2F\$a\$S^2-\{(E+F)^2-2S^2\}\$aS_C\$$ $+\$aS_C^3\$]/\$aS_A\$$	
X(523)	0	$(E-8F)S^2/(E+F)$	de Long-champs(A)
X(524)	$\{(E+F)^2-3S^2\}/(E+F)$	$3\$S_A^2(S_B-S_C)\$/\{2(E+F)\}$	GK(L)
X(525)	$\$S_A^2(S_B-S_C)\$/S^2$	$-2\{3(E+F)F-S^2\}$	
X(526)	$-4\$S_A^2(S_B-S_C)\$$ $/\{(E+F)(E-5F)+S^2\}$	$\{(E^2+20EF-8F^2)-8S^2\}S^2$ $/\{(E+F)(E-5F)+S^2\}$	
X(527)	$-\{3(E+F)^2-6S^2+(E+F)\$ab\$$ $-3\$abS_C\$\}/[2\{2(E+F)+\$ab\$\}]$	$-3\{\$S_A^2(S_B-S_C)\$$ $-\$abS_C(S_A-S_B)\$\}$ $/[2\{(E+F)+\$ab\$\}]$	X(2)X(7)(L)
X(528)	$[(E-8F)S^2-abc\{(E+F)\$a\$$ $-3\$aS_A\$\}]/\{3(S^2+\$a\$abc)\}$	$abc\$aS_A(S_B-S_C)\$/(S^2+\$a\$abc)$	X(2)X(11)(L)
X(529)	$-[(E-8F)S^2+abc\{(E+F)\$a\$$ $-3\$aS_A\$\}]/\{3(S^2+\$a\$abc)\}$	$abc\$aS_A(S_B-S_C)\$/(S^2+\$a\$abc)$	X(2)X(12)(L)
X(530)	$-[2(E+F)^2-6S^2+(3)^{1/2}(E-8F)S]$ $/[3\{2(E+F)+(3)^{1/2}S\}]$	$-\$S_A^2(S_B-S_C)\$$ $/\{2(E+F)+(3)^{1/2}S\}$	X(2)X(13)(L)
X(531)	$[2(E+F)^2+6S^2-(3)^{1/2}(E-8F)S]$	$\$S_A^2(S_B-S_C)\$$	X(2)X(14)(L)

	$/[3\{2(E+F)+(3)^{1/2}S\}]$	$/\{2(E+F)+(3)^{1/2}S\}$	
X(532)	$-[6\{(E+F)^2-3S^2\}+(3)^{1/2}(E-8F)S]$ $/[3\{3(E+F)+(3)^{1/2}S\}]$	$-3\$S_A^2(S_B-S_C)\$$ $/\{3(E+F)+(3)^{1/2}S\}$	X(2)X(17)(L)
X(533)	$[6\{(E+F)^2-3S^2\}-(3)^{1/2}(E-8F)S]$ $/[3\{3(E+F)+(3)^{1/2}S\}]$	$3\$S_A^2(S_B-S_C)\$$ $/\{3(E+F)+(3)^{1/2}S\}$	X(2)X(18)(L)
X(534)	$\{3\$a\$FS^2-(E+F)\$aS_BS_C\$\}$ $/(2\$aS_BS_C\$)$	$3\$a(S_B-S_C)\$FS^2/\{2\$aS_BS_C\$\}$	X(2)X(19)(L)
X(535)	$-[2(E-8F)S^2+abc\{(E+F)\$a\$$ $-3\$aS_A\$\}]/[3(\$a\$abc+2S^2)]$	$abc\$aS_A(S_B-S_C)\$]/(\$a\$abc$ $+2S^2)$	X(2)X(36)(L)
X(536)	$-\{(E+F)\$ab\$-3\$abS_C\$\}/(2\$ab\$)$	$3\$abS_C(S_A-S_B)\$/(2\$ab\$)$	X(2)X(37)(L)
X(537)	$-\{(E+F)^2\$a\$-2(E+F)\$aS_A\$$ $-3\$aS_A^2\$\}$ $/[2\{(E+F)\$a\$+\$aS_A\$\}]$	$3\{(E+F)\$aS_A(S_B-S_C)\$$ $+\$aS_A^2(S_B-S_C)\$\}$ $/[2\{(E+F)\$a\$+\$aS_A\$\}]$	X(2)X(38)(L)
X(538)	$-\{2(E+F)^3-(7E-2F)S^2\}$ $/\{2\{(E+F)^2-2S^2\}$	$-3(E+F)\$S_A^2(S_B-S_C)\$$ $/\{2\{(E+F)^2-2S^2\}$	X(2)X(39)(L)
X(539)	$\{(3E^2+14EF+20F^2)-12S^2\}$ $/\{3(3E+4F)\}$	$2\$S_A^2(S_B-S_C)\$/(3E+4F)$	X(2)X(54)(L)
X(540)	$-[2(E+F)^3-(5E+14F)S^2$ $+2\{(E+F)^2-3S^2\}\$ab\$]$ $/[2\{(E+F)\$a\$^2-2S^2\}]$	$-3\$S_A^2(S_B-S_C)\$\{(E+F)$ $+\$\$ab\$\}/[2\{(E+F)\$a\$^2-2S^2\}]$	X(2)X(58)(L)
X(541)	$\{(E^2+38EF-44F^2)-12S^2\}$ $/\{3(5E+14F)\}$	$2\$S_A^2(S_B-S_C)\$/(5E+14F)$	X(2)X(74)(L)
X(542)	$-\{(E+F)(E+10F)-6S^2\}/\{6(E+F)\}$	$-\$S_A^2(S_B-S_C)\$/\{2(E+F)\}$	Fermat(L)
X(543)	$\{2(E+F)^3+3(E-2F)S^2\}$ $/\{3(E+F)^2\}$	$\$S_A^2(S_B-S_C)\$/(E+F)$	X(2)X(99)(L)
X(544)	$\{2(E+F)^3-(5E+14F)S^2+\$ab\$S^2$ $-3(E+F)\$abS_C\$+\$abS_C^2\$$ $+5\$abS_AS_B\$\}/\{4(E+F)^2-4S^2$ $+2(E+F)\$ab\$+abc\$a\$\}$	$3\{(E+F)\$S_A^2(S_B-S_C)\$$ $-(E+F)\$abS_C(S_A-S_B)\$$ $-\$abS_AS_B(S_A-S_B)\$\}$ $/\{4(E+F)^2-4S^2+2(E+F)\$ab\$$ $+abc\$a\$\}$	X(2)X(101)(L)
X(545)	$-\{(E+F)^2-3S^2+(E+F)\$ab\$$ $-3\$abS_C\$\}/\$a\$^2$	$-3\{\$S_A^2(S_B-S_C)\$$ $-2\$abS_C(S_A-S_B)\$\}/(2\$a\$^2)$	X(2)X(45)(L)